Mesopotamian Mathematics: Some Historical Background

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Introduction

> When I was young I learned at school the scribal art on the tablets of Sumer and Akkad. Among the high-born no-one could write like me. Where people go for instruction in the scribal art there I mastered completely subtraction, addition, calculation and accounting.[1]

Most mathematicians know at least a little about 'Babylonian' mathematics: about the sexagesimal place value system, written in a strange wedge-shaped script called cuneiform; about the very accurate approximation to $\sqrt{2}$;[2] and about the famous list of Pythagorean triples, Plimpton 322.[3] This kind of information is in most math history books. So the aim of this article is not to tell you about things which you can easily read about elsewhere, but to provide a context for that mathematics—a brief overview of nearly five millennia of mathematical development and the environmental and societal forces which shaped those changes.[4]

So where are we talking about, and when? The Greek word 'Mesopotamia' means 'between the rivers' and has referred to the land around the Tigris and Euphrates in modern day Iraq since its conquest by Alexander the Great in 330 BCE. But its history goes back a good deal further than that. Mesopotamia was settled from the surrounding hills and mountains during the course of the fifth millennium BCE. It was here that the first sophisticated, urban societies grew up, and here that writing was invented, at the end of the fourth millennium, perhaps in the southern city of Uruk. Indeed, writing arose directly from the need to record mathematics and accounting: this is the subject of the first part of the article. As the third millennium wore on, counting and measuring systems were gradually revised in response to the demands of large-scale state bureaucracies. As the second section shows, this led in the end to the sexagesimal, or base 60, place value system (from which the modern system of counting hours, minutes and seconds is ultimately derived).

By the beginning of the second millennium, mathematics had gone beyond the simply utilitarian. This period produced what most of the text-books call 'Babylonian' mathematics, although, ironically, it is highly unlikely that any of the math comes from Babylon itself: the early second millennium city is now deep under the water table and impossible to excavate. The third part of this article examines the documents written in the scribal schools to look for evidence of how math was taught at this time, and why it might have moved so far from its origins. But after the mid-second millennium BCE we have almost no knowledge of mathematical activity in Mesopotamia, until the era of the Greek conquest in the late fourth century BCE—when math from the city of Babylon is known. The fourth and final part looks at why there is this enormous gap in the record: was there really very little math going on, or can we find some other explanations for our lack of evidence?

Counting with clay: from tokens to tablets

But now let us start at the beginning. The Tigris-Euphrates valley was first inhabited during the mid-fifth millennium BCE. Peoples who had already been farming the surrounding hills of the so-called 'Fertile Crescent' for two or three millennia began to settle, first in small villages, and then in increasingly large and sophisticated urban centres. The largest and most complex of these cities were Uruk on the Euphrates, and Susa on the Shaur river. Exactly why this urban revolution took place need not concern us here; more important to the history of mathematics are the consequences of that enormous shift in societal organisation.

Although the soil was fertile and the rivers full, there were two major environmental disadvantages to living in the southern Mesopotamian plain. First, the annual rainfall was not high enough to support crops without artificial irrigation systems, which were in turn vulnerable to destruction when the rivers flooded violently during each spring harvest. Second, the area yielded a very limited range of natural resources: no metals, minerals, stones or hard timber; just water, mud, reeds and date-palms. Other raw materials had to be imported, by trade or conquest, utilised sparingly, and recycled. So mud and reeds were the materials of everyday life: houses and indeed whole cities were made of mud brick and reeds; the irrigation canals and their banks were made of mud reinforced with reeds; and there were even some experiments in producing agricultural tools such as sickles from fired clay.

It is not surprising then that mud and reeds determined the technologies available for other everyday activities of urban society, such as managing and monitoring labour and commodities. The earliest known method of controlling the flow of goods seems to have been in operation from the time of the earliest Mesopotamian settlement, predating the development of writing by millennia [Nissen, Damerow and Englund 1993: 11]. It used small clay 'tokens' or 'counters', made into various geometric or regular shapes. Each 'counter' had both quantitative and qualitative symbolism: it represented a specific number of a certain item. In other words it was not just a case of simple one-to-one correspondence: standard groups or quantities could also be represented by a single token. It is often impossible to identify exactly which commodity a particular token might have depicted; indeed, when such objects are found on their own or in ambiguous contexts, it is rarely certain whether they were used for accounting at all. The clearest evidence comes when these tokens are found in round clay envelopes, or 'bullae', whose surfaces are covered in impressed patterns. These marks were made, with an official's personal cylinder seal, to prevent tampering. The envelope could not be opened and tokens removed without damaging the pattern of the seal. In such a society, in which literacy was restricted to the professional few, these cylinder-seals were a crucial way of marking individual responsibility or ownership and, like the tokens, are ideally suited to the medium of clay.

Of course, sealing the token-filled envelopes meant that it was impossible to check on their contents, even legitimately, without opening the envelope in the presence of the sealing official. This problem was overcome by impressing the tokens into the clay of the envelope before they were put inside. It then took little imagination to see that one could do without the envelopes altogether. A deep impression of the tokens on a piece of clay, which could also be sealed by an official, was record enough.

At this stage, c. 3200 BCE, we are still dealing with tokens or their impressions which represent both a number and an object in one. A further development saw the separation of the counting system and the objects being counted. Presumably this came about as the range of goods under central control widened, and it became unfeasible to create whole new sets of number signs each time a new commodity was introduced into the accounting system. While we see the continuation of *impressions* for numbers, the objects themselves were now represented on clay either by a drawing of the object itself or of the token it represented, *incised* with a sharp reed. Writing had begun.[5]

Now mathematical operations such as arithmetic could be recorded. The commodities being counted cannot usually be identified, as the incised signs which represent them have not yet been deciphered. But the numerals themselves, recorded with impressed signs, can be identified with ease. For instance, one tablet displays a total of eighteen D-shaped marks on the front, and three round ones, in four separate enclosures. On the back are eight Ds and four circles, in one enclosure.[6] We can conclude that the circular signs must each be equivalent to ten Ds. In fact, we know from other examples that these two signs do indeed represent 1 and 10 units respectively, and were used for counting discrete objects such as people or sheep.

Using methods like this, a team in Berlin have identified a dozen or more different systems used on the ancient tablets from Uruk [Nissen, Damerow and Englund, 1993:

28–29]. There were four sets of units for counting different sorts of discrete objects, another set for area measures, and another for counting days, months and years. There were also four capacity measure systems for particular types of grain (apparently barley, malt, emmer and groats) and two for various kinds of dairy fat. A further system is not yet completely understood; it may have recorded weights. Each counting or measuring system was context-dependent: different number bases were used in different situations, although the identical number signs could be used in different relations within those contexts. One of the discrete-object systems was later developed into the sexagesimal place value system, while some of the other bases were retained in the relationships between various metrological units. It is an enormously complex system, which has taken many years and a lot of computer power to decipher; the project is still unfinished.

It is unclear what language the written signs represent (if indeed they are language-specific), but the best guess is Sumerian, which was certainly the language of the succeeding stages of writing. But that's another story; it's enough for our purposes to see that the need to record number and mathematical operations efficiently drove the evolution of recording systems until one day, just before 3000 BCE, someone put reed to clay and started to write mathematics.

The third millennium: math for bureaucrats

During the course of the third millennium writing began to be used in a much wider range of contexts, though administration and bureaucracy remained the main function of literacy and numeracy. This restriction greatly hampers our understanding of the political history of the time, although we can give a rough sketch of its structure. Mesopotamia was controlled by numerous city states, each with its own ruler and city god, whose territories were concentrated on the canals which supplied their water. Because the incline of the Mesopotamian plain is so slight—it falls only around 5 cm in every kilometre—large-scale irrigation works had to feed off the natural watercourses many miles upstream of the settlements they served. Violent floods during each year's spring harvest meant that their upkeep required an enormous annual expenditure. The management of both materials and labour was essential, and quantity surveying is attested prominently in the surviving tablets.

Scribes had to be trained for their work and, indeed, even from the very earliest phases around 15% of the tablets discovered are standardised practice lists—of titles and professions, geographical names, other sorts of technical terminology. From around 2500 BCE onwards such 'school' tablets—documents written for practice and not for working use—include some mathematical exercises. By this time writing was no longer restricted to nouns and numbers. By using the written signs to represent the *sounds* of the objects they represented and not the objects themselves, scribes were able to record other parts of human speech, and from this we know that the earliest school math was written in a now long-dead language called Sumerian. We currently have a total of about thirty mathematical tablets from three mid-third millennium cities—Shuruppak, Adab and Ebla—but there is no reason to suppose that they represent the full extent of mathematical knowledge at that time. Because it is often difficult to distinguish between competently written model documents and genuine archival texts, many unrecognised school tablets, from all periods, must have been published classified as administrative material.

Some of the tablets from Shuruppak state a single problem and give the numerical answer below it [Powell, 1976: 436 n19]. There is no working shown on the tablets, but these are more than simple practical exercises. They use a practical pretext to explore the division properties of the so-called 'remarkable numbers' such as 7, 11, 13, 17 and 19, which are both irregular (having factors other than 2, 3 and 5) and prime [cf. Høyrup, 1993]. We also have a geometrical diagram on a round tablet from Shuruppak and two contemporary tables of squares from Shuruppak and Adab which display consciously sexagesimal characteristics [Powell, 1976: 431 & fig. 2]. The contents of the tablets from Ebla are more controversial: according to one interpretation, they contain metrological tables which were used in grain distribution calculations [Friberg, 1986].

Mesopotamia was first unified under a dynasty of kings based at the undiscovered city of Akkad, in the late twenty-fourth century BCE. During this time the traditional metrological systems were overhauled and linked together, with new units based on divisions of sixty. Brick sizes and weights were standardised too [Powell, 1987–90: 458]. The new scheme worked so well that it was not substantially revised until the mid-second millennium, some 800 years later; indeed, as we shall see, some Akkadian brick sizes were still being used in the Greek period, in the late fourth century BCE.

There are only eight known tablets containing mathematical problems from the Akkadian period, from Girsu and Nippur. The exercises concern squares and rectangles. They either consist of the statement of a single problem and its numerical answer, or contain two stated problems which are allocated to named students. In these cases the answers are not given, and they appear to have been written by an instructor in preparation for teaching. Indeed, one of these assigned problems has a solved counterpart amongst the problem texts. Certain numerical errors suggest that the sexagesimal place system was in use for calculations, at least in prototype form [Whiting, 1984].

A round tablet from Nippur shows a mathematical diagram which displays a concern with the construction of problems to produce integer solutions. The trapezoid has a transversal line parallel to the base, dividing it into two parts of equal area. The lengths of the sides are chosen in such a way that the length of the transversal line can be expressed in whole numbers [Friberg, 1987–90: 541]. No mathematical tables are known from this period, but model documents of various kinds have been identified, including a practice account from Eshnunna and several land surveys and building plans [Westenholz, 1977: 100 no. 11; Foster, 1982: 239–40]. In working documents too, we see a more sophisticated approach to construction and labour management, based on the new metrological systems. The aim was to predict not only the raw materials but also the manpower needed to complete state-funded agricultural, irrigation and construction projects, an aim which was realised at the close of the millennium under the Third Dynasty of Ur.

The Ur III empire began to expand rapidly towards the east in the second quarter of the 21st century BCE. At its widest extent it stretched to the foothills of the Zagros mountains, encompassing the cities of Urbilum, Ashur, Eshnunna and Susa. To cope with the upkeep of these new territories and the vastly increased taxation revenues they brought in, large-scale administrative and economic reforms were executed over the same period. They produced a highly centralised bureaucratic state, with virtually every aspect of its economic life subordinated to the overriding objective of the maximisation of gains. These administrative innovations included the creation of an enormous bureaucratic apparatus, as well as of a system of scribal schools that provided highly uniform scribal and administrative training for the prospective members of the bureaucracy. Although little is currently known of Ur III scribal education, a high degree of uniformity must have been essential to produce such wholesale standardisation in the bureaucratic system.

As yet only a few school mathematical texts can be dated with any certainty to the Ur III period, but between them they reveal a good deal about contemporary educational practice. There are two serious obstacles to the confident identification of school texts from the Ur III period when, as is often the case, they are neither dated nor excavated from well-defined find-spots. Firstly, there is the usual problem of distinguishing between competently written practice documents and those produced by working scribes. Secondly, palaeographic criteria must be used to assign a period to them. In many cases it is matter of dispute whether a text is from the late third millennium or was written using archaising script in the early second millennium. In particular, it was long thought that the sexagesimal place system, which represents numerals using just tens and units signs, was an innovation of the following Old Babylonian period so that any text using that notation was assumed to date from the early second millennium or later. However, we now know that it was already in use by around 2050 BCE—and that the conceptual framework for it had been under construction for several hundred years. Crucially, though, calculations in sexagesimal notation were made on temporary tablets which were then reused after the calculation had been transferred to an archival document in standard notation [Powell, 1976: 421].[7] We should expect, then, to find neither administrative documents using the sexagesimal system nor sexagesimal school texts which were used to train the scribes (because, in general, they were destroyed after use, and we can hardly distinguish them from later examples).

One conspicuous exception to our expectations is a round model document from Girsu [Friberg, 1987–90: 541]. On one side of the tablet is a (slightly incorrect) model entry from a quantity survey, giving the dimensions of a wall and the number of bricks in it. The measurements of the wall are given in standard metrological units, but have been (mis-)copied on to the reverse in sexagesimal notation. The volume of the wall, and the number of bricks in it, are then worked out using the sexagesimal numeration, and converted back into standard volume and area measure, in which systems they are written on the obverse of the tablet. These conversions were presumably facilitated by the use of metrological tables similar to the many thousands of Old Babylonian exemplars known. In other words, scribal students were already in the Ur III period taught to perform their calculations—in sexagesimal notation—on tablets separate from the model documents to which they pertained, which were written in the ubiquitous mixed system of notation.

The writer of that tablet from Girsu might easily have gone on to calculate the labour required to make the bricks, to carry them to the building site, to mix the mortar, and to construct the wall itself. These standard assumptions about work rates were at the heart of the Ur III regime's bureaucracy. Surveyors' estimates of a work gang's expected outputs were kept alongside records of their actual performances—for tasks as diverse as milling flour to clearing fallow fields. At the end of each administrative year, accounts were drawn up, summarising the expected and true productivity of each team. In cases of shortfall, the foreman was responsible for catching up the following year; but work credits could not be carried over [Englund, 1991]. The constants used in these administrative calculations are found in a few contemporary school practice texts too [Robson 1999: 31].

Math education in the early second millennium

But such a totalitarian centrally-controlled economy could not last, and within a century the Ur III empire had collapsed under the weight of its own bureaucracy. The dawn of the second millennium BCE—the so-called Old Babylonian period—saw the rebirth of the small city states, much as had existed centuries before. But now many of the economic functions of the central administration were deregulated and contracted out to private enterprise. Numerate scribes were still in demand, though, and we have an unprecedented quantity of tablets giving direct or indirect information on their training. Many thousands of school tablets survive although they are for the most part unprovenanced, having been dug up at the end of the nineteenth century (CE!) before the advent of scientific archaeology. However, mathematical tablets have been properly excavated from a dozen or so sites, from Mari and Terqa by the Euphrates on the Syria-Iraq border to Me-Turnat on the Diyala river and Susa in south-west Iran.

We know of several school houses from the Old Babylonian period, from southern Iraq [Stone, 1987: 56–59; Charpin, 1986: 419–33]. They typically consist of several small rooms off a central courtyard, and would be indistinguishable from the neighbouring dwellings if it were not for some of the fittings and the tablets that were found inside them. The courtyard of one house in Nippur, for instance, had built-in benches along one side and a large fitted basin containing a large jug and several small bowls which are thought to have been used for the preparation and moistening of tablets. There was also a large pile of crumpled up, half-recycled tablets waiting for re-use. The room behind the courtyard had been the tablet store, where over a thousand school tablets had been shelved on benches and perhaps filed in baskets too. Judging by the archaeological evidence and the dates on some of the tablets, both school houses were abandoned suddenly during the political upheavals of 1739 BCE. If the buildings had fallen into disuse or their functions had changed for more peaceful reasons, we would expect the tablets to have been cleared out of the houses, or perhaps used as rubble in rebuilding work.

Some of the school tablets were written by the teachers, while others were 'exercise tablets' composed by the apprentice scribes. Sumerian, which had been the official written language of the Ur III state, was gradually ousted by Akkadian—a Semitic language related to Hebrew and Arabic but which used the same cuneiform script as Sumerian. Akkadian began to be used for most everyday writings while Sumerian was reserved for scholarly and religious texts, analogous to the use of Latin in Europe until very recently. This meant that much of the scribal training which had traditionally been oral was recorded in clay for the first time, either in its original Sumerian, or in Akkadian translation, as was the case for the mathematical texts.

Math was part of a curriculum which also included Sumerian grammar and literature, as well as practice in writing the sorts of tablets that working scribes would need. These included letters, legal contracts and various types of business records, as well as more mathematically oriented *model documents* such as accounts, land surveys and house plans. Five further types of school mathematical text have been identified, each of which served a separate pedagogical function [Robson, 1999: 8–15]. Each type has antecedents in the third millennium tablets discussed in the previous section.

First, students wrote out *tables* while memorising metrological and arithmetical relationships. There was a standard set of multiplication tables, as well as aids for division, finding squares and square roots, and for converting between units of measurement. Many scribes made copies for use at work too. *Calculations* were carried out, in formal layouts, on small round tablets—called 'hand tablets'—very like the third millennium examples mentioned above. Hand tablets could serve as the scribes' 'scratch pads' and might also carry diagrams and short notes as well as handwriting practice and extracts from literature. The teacher set mathematical problems from 'textbooks'—usually called *problem* texts in the modern literature—which consisted of a series of (often minimally different) problems and their numerical answers. They might also contain model solutions and diagrams. Students sometimes copied problem texts, but they were for the most part composed and transmitted by the scribal teachers. Teachers also kept *solution lists* containing alternative sets of parameters, all of which would give integer answers for individual problems [Friberg, 1981]. There were also tables of technical constants—conventionally known as *coefficient lists*—many of whose entries are numerically identical to the constants used by the personnel managers of the Ur III state [Kilmer, 1960; Robson, 1999].[8]

Model solutions, in the form of algorithmic instructions, were not only didactically similar to other types of educational text, but were also intrinsic to the very way mathematics was conceptualised. For instance, the problems which have conventionally been classified as 'quadratic equations' have recently turned out to be concerned with a sort of cut-and-paste geometry [Høyrup, 1990; 1995]. As the student followed the instructions of the model solution, it would have been clear that the method was right—because it worked—so that no proof was actually needed.

The bottom line for Old Babylonian education must have been to produce literate and numerate scribes, but those students were also instilled with the aesthetic pleasure of mathematics for its own sake. Although many ostensibly practical scenarios were used as a pretext for setting non-utilitarian problems, and often involved Ur III-style technical constants, they had little concern with accurate mathematical modelling. Let us take the topic of grain-piles as an example. In the first sixteen problems of a problem text from Sippar the measurements of the grain-pile remain the same, while each parameter is calculated in turn.[9] The first few problems are missing, but judging from other texts we would expect them to be on finding the length, then the width, height, etc. The first preserved problem concerns finding the volume of the top half of the pile.

One could imagine how such techniques might be useful to a surveyor making the first estimate of the capacity of a grain-pile after harvest—and indeed we know indirectly of similar late third millennium measuring practices. However, then things start to get complicated. The remaining problems give data such as the sum of the length and top, or the difference between the length and the thickness, or even the statement that the width is equal to half of the length plus 1. It is hardly likely that an agricultural overseer would ever find himself needing to solve this sort of a problem in the course of a working day.

Similarly, although the mathematical grain-pile is a realistic shape—a rectangular pyramid with an elongated apex—even simply calculating its volume involves some rather sophisticated three-dimensional geometry, at the cutting edge of Old Babylonian mathematics as we know it. Further, it appears that at some point the scenario was further refined to enable mathematically more elegant solutions to be used in a tablet from Susa.[10] In both sets of problems the pile is 60 m long and 18–24 m high. It is difficult to imagine how a grain pile this big could ever be constructed, let alone measured with a stick. In short, the accurate mathematical modelling of the real world was not a priority of Old Babylonian mathematics; rather it was concerned with approximations to it that were both good enough and mathematically pleasing.

The evidence for mathematical methods in the Old Babylonian workplace is still sketchy, but one can look for it, for instance, in canal and land surveys. Although these look rather different from their late third millennium precursors—they are laid out in the form of tables, with the length, width and depth of each excavation in a separate column, instead of in lists—the mathematical principles involved are essentially the same. There is one important distinction though; there is no evidence (as yet) for work-rate calculations. This is not surprising; we are not dealing with a centralised 'national' bureaucracy in the early second millennium, but quasi-market economies in which much of the work traditionally managed by the state was often contracted out to private firms bound by legal agreements. One would not expect a consistent picture of quantitative management practices throughout Mesopotamia, even where such activities were documented.

What happened next?
Tracing the path to Hellenistic Babylon

After about 1600 BCE mathematical activity appears to come to an abrupt halt in and around Mesopotamia. Can it simply be that math was no longer written down, or can we find some other explanation for the missing evidence?

For a start, it should be said that there is a sudden lack of tablets of all kinds, not just school mathematics. The middle of the second millennium BCE was a turbulent time, with large population movements and much political and social upheaval. This must have adversely affected the educational situation. But there is the added complication that few sites of this period have been dug, and that further, the tablets which have been excavated have been studied very little. Few scholars have been interested in this period of history, partly because the documents it has left are so difficult to decipher.

But, further, from the twelfth century BCE onwards the Aramaic language began to take over from Akkadian as the everyday vehicle of both written and oral communication. Aramaic was from the same language-family as Akkadian, but had adopted a new technology. It was written in ink on various perishable materials, using an alphabet instead of the old system of syllables on clay. Sumerian, Akkadian and the cuneiform script were retained for a much more restricted set of uses, and it may be that math was not usually one of them. It appears too that cuneiform was starting to be written in another new medium, wax-covered ivory or wooden writing-boards, which could be melted down and smoothed off as necessary. Although contemporary illustrations and references on clay tablets indicate that these boards were in widespread use, very few have been recovered—all in watery contexts which aided their preservation—but the wax had long since disappeared from their surfaces. So even if mathematics were still written in cuneiform, it might well have been on objects which have not survived.

These factors of history, preservation and fashions in modern scholarship have combined to mean that the period between around 1600 and 1000 BCE in south Mesopotamia is still a veritable dark age for us. The light is beginning to dawn, though, and there is no reason why school texts, including mathematics, should not start to be identified, supposing that they are there to be spotted. But, fortunately for us, the art of writing on clay did not entirely die out, and there are a few clues available already. Mathematical and metrological tables continued to be copied and learnt by apprentice scribes; they have been found as far afield as Ashur on the Tigris, Haft Tepe in southwest Iran, and Ugarit, Hazor and Byblos on the Mediterranean coast. One also finds evidence of non-literate mathematical concepts, which have a distinctly traditional flavour. Not only do brick sizes remain more or less constant—which strongly suggests that some aspects of third millennium metrology were still in use—but there are also some beautiful and sophisticated examples of geometrical decoration. There are, for instance, stunning patterned ‘carpets’ carved in stone from eighth and seventh century Neo-Assyrian palaces—an empire more renowned for its brutal deportations and obsession with astrology than for its contributions to cultural heritage.

But perhaps more excitingly, a mathematical problem is known in no less than three different copies, from Nineveh and Nippur.[11] Multiple exemplars are rare in the mathematically-rich Old Babylonian period, but for the barren aftermath it may be an indication of the reduced repertoire of problems in circulation at that time. Its style shows that mathematical traditions of the early second millennium had not died out, while apparently new scenarios for setting problems had developed. It is a teacher’s problem text, for a student to solve, and it is couched in exactly the sort of language known from the Old Babylonian period. But interestingly it uses a new pretext. The problem ostensibly concerns distances between the stars, though in fact it is about dealing with division by ‘remarkable’ numbers—a topic which, as we have seen, goes back as far as the mid-third millennium.

Finally we arrive in Babylon itself—a little later than the Persians and Greeks did. By the fourth and third centuries BCE indigenous Mesopotamian civilisation was dying. Some of the large merchant families of Uruk and Babylon still used tablets to record their transactions, but the temple libraries were the principal keepers of traditional cuneiform culture. Their collections included huge series of omens, historical chronicles, and mythological and religious literature as well as records of astronomical observations. It has often been said that mathematics by now consisted entirely of mathematical methods for astronomy, but that is not strictly true. As well as the mathematical tables—now much lengthier and sophisticated than in earlier times—we know of at least half a dozen tablets containing non-astronomical mathematical problems for solution. Although the terminology and conceptualisation has changed since Old Babylonian times—which, after all, is only to be expected—the topics and phraseology clearly belong to the same stream of tradition. Most excitingly, a small fragment of a table of technical constants has recently been discovered, which contains a list of brick sizes and densities. Although the mathematics involved is rather more complicated than that in similar earlier texts, the brick sizes themselves are exactly identical to those invented in the reforms of Akkad around two thousand years before.

Conclusions

I hope I have been able to give you a little taste of the rich variety of Mesopotamian math that has come down to us. Its period of development is vast. There is twice the timespan between the first identifiable accounting tokens and the latest known cuneiform mathematical tablet as there is between that tablet and this book. Most crucially, though, I hope that you will agree with me that mathematics is fundamentally a product of society. Its history is made immeasurably richer by the study of the cultures which have produced it, wherever and whenever they might be.

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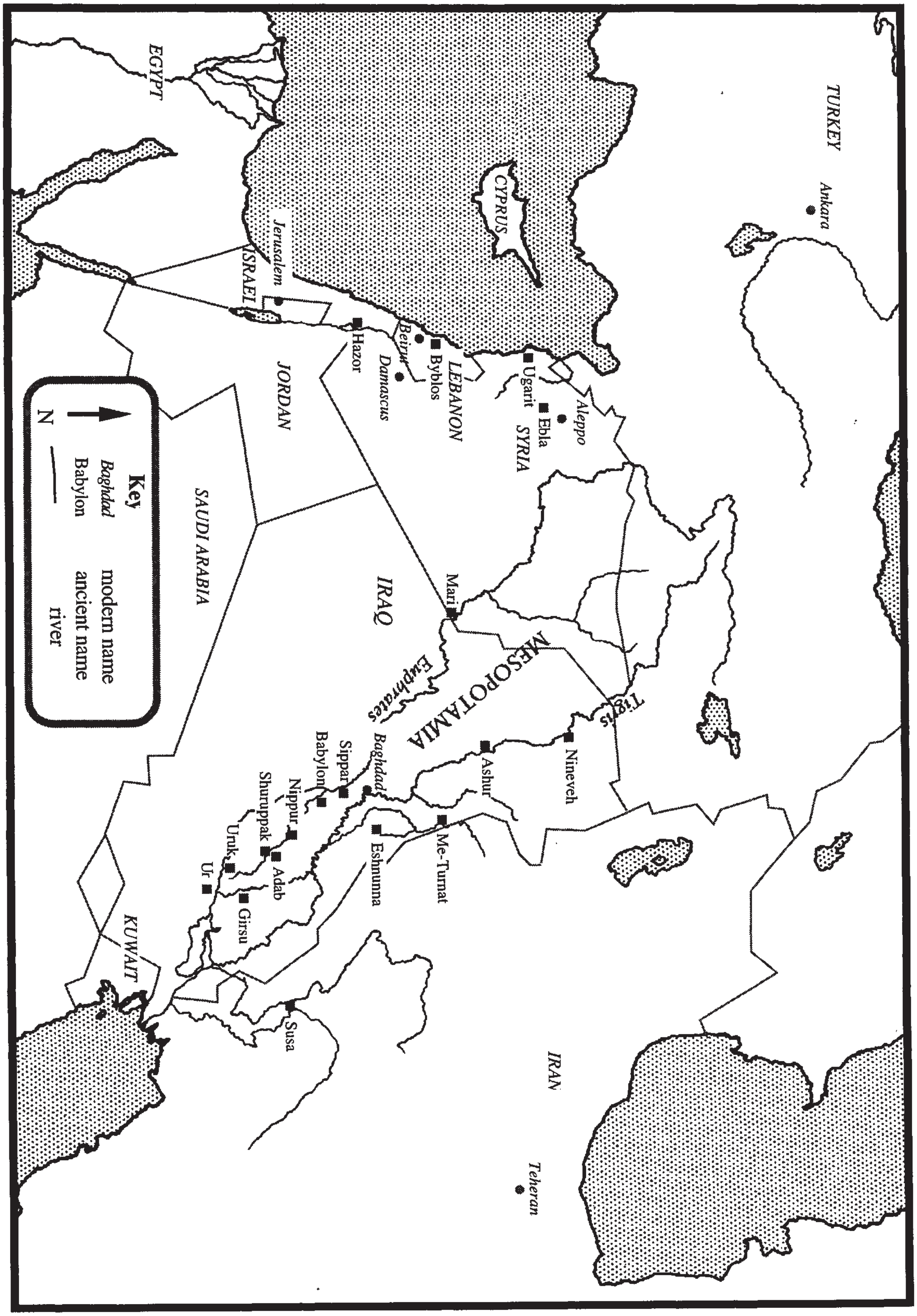
	POLITICAL PERIODISATION	MATHEMATICAL DEVELOPMENTS	SOCIETY AND TECHNOLOGY	THE REST OF THE WORLD
IRON AGE (0 AD/BC – 1000 BC); *Late Babylonian Period*	*Parthian (Arsacid) Period 126 Bc–227 AD* *Seleucid (Helenistic, or Greek) period, 330–127* *Persian (Achaemenid) empire, 538–331* *Neo-Babylonian empire, 625–539* *Neo-Assyrian empire, 883–612*	Latest known cuneiform tablets are astronomical records Math and astronomy maintained and developed by temple personnel Mathematical tradition apparently continues, although the evidence is currently very slight	*Traditional Mesopotamian culture dying under the influence of foreign rulers* *Cotton* *Coinage* *Brass* *Cuneiform Akkadian being replaced by alphabetic Aramaic*	Invention of paper, in China Great Wall of China, 214 The *Elements* Birth of Buddha, c. 570 Foundation of Rome, c. 750 Indian mathematics
LATE BRONZE	*Middle Babylonian period, c. 1150–626* *Kassite period, c. 1600–1150*	A few mathematical tables known, from sites on the periphery of Mesopotamia	*Smelted iron, Camels* *Glazed pottery, Glass*	Tutankhamen
MIDDLE BRONZE (2000 BC)	*Old Babylonian period, c. 2000–1600*	Best-documented period of math in scribal schools	*Horse and wheel technology improved* *Much Sumerian and Akkadian literature*	Stonehenge completed Rhind mathematical papyrus Earliest recorded eclipse: China, 1876
EARLY BRONZE AGE (2000 BC – 3000 BC)	*Ur III empire, c. 2100–2000* *Kingdon of Akkad, c. 2350–2150* *Early Dynastic period, c. 3000–2350*	Development of the sexagesimal place value system Reform of the metrological systems Earliest known math tables	*First large empires* *Ziqqurats* *Horses* *Akkadian written in cuneiform characters* *City states* *Palaces* *Development of writing into cuneiform (Sumerian)* *High Sumerian culture*	Indus Valley civilisation Collapse of the Egyptian Old Kingdom Upper and Lower Egypt united Great Pyramid Stonehenge begun
(3000 BC – 4000 BC)	*Uruk period, c. 4000–3000*	Earliest known written documents: accounts using complex metrological systems Development of clay token accounting system, with sealed 'bullae'	*URBANISATION* *Beginnings of writing and bureaucracy* *Cylinder seals* *Monumental architecture* *Potter's wheel* *Bronze, gold and silver work*	
NEOLITHIC AGE (4000 BC – 5000 BC)	*Ubaid period, c. 5500–4000*	Small, regularly-shaped clay 'tokens' apparently used as accounting devices	*Irrigation agriculture* *Wide use of brick* *Temples* *Copper and pottery* *Mesopotamia begins to be settled by farmers from the surrounding hills (the Fertile Crescent)*	Megalithic cultures of western Europe First temple towers in South America Farming begins to reach Europe from the Near East

Time chart showing major political, societal, technological and mathematical developments in the ancient Near East.[12]



Map showing the principal modern cities of the Near East and all the ancient sites mentioned in the text.

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Endnotes

[1] From a hymn of self-praise to king Shulgi, 21st century BCE; cf. Castellino 1972: 32.

[2] 1;24 51 10 ($\approx 1.4142129\ldots$) in YBC 7289 [Neugebauer and Sachs 1945: 42]. In general I have tried to cite the most recent, reliable and easily accessible sources, rather than present an exhaustive bibliography for the topic.

[3] See, for instance, Joseph, 1991: 91–118; Katz, 1993: 6–7, 24–28.

[4] For general works on ancient Near Eastern history and culture, see the suggestions for further reading at the end.

[5] According to a recent theory, tokens could have been used like abacus counters for various arithmetical operations [Powell 1995].

[6] VAT 14942: see Nissen, Damerow and Englund, 1993: pl. 22.

[7] That is, in cuneiform signs which indicate both the absolute value of the number and the system of measurement used.

[8] The major publications of Old Babylonian mathematical texts are still Neugebauer, 1935–37; Thureau-Dangin, 1938; Neugebauer and Sachs, 1945; Bruins and Rutten, 1961. For an index of more recent publications, editions and commentaries, see Nemet-Nejat, 1993.

[9] BM 96954 + BM 102366 + SÉ 93, published in Robson, 1999: Appx. 3.

[10] TMS 14; Robson, 1999: ch. 7.

[11] HS 245, Sm 162, Sm 1113. See most recently Horowitz, 1993.

[12] Dates earlier than 911 BCE are not accurate, and vary from book to book and scholar to scholar, as do the names and dates of the periods into which Mesopotamian political history is conventionally divided.